PERGAMON

International Journal of Multiphase Flow 25 (1999) 705–713

International Journal of Multiphase Flow

The force on a growing bubble in potential flow

J.D. Sherwood [1]

Schlumberger Cambridge Research, High Cross, Madingley Road, Cambridge, CB3 0EL, UK

Received 29 August 1997; received in revised form 14 September 1998

Abstract

Zuber's analysis of the acceleration reaction on a bubble in a concentrated suspension is extended to the case in which the bubble radius changes with time. The analysis is based on a cell model: the bubble is at the centre of a sphere filled with incompressible fluid. The bubble grows and translates, whereas the centre of the outer boundary is at rest; thus the calculations are performed in a frame of zero total volume flux. The predicted force on the bubble in a concentrated suspension agrees, in the dilute limit, with previous results for the force on a single bubble growing in unbounded fluid, and becomes large as the volume fraction of the gas increases. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: Added mass; Bubble; Compressible flow; Potential flow

1. Introduction

The added mass of an accelerating bubble is usually explained in terms of the inertia of the surrounding, entrained liquid which must also accelerate. The volume of liquid so entrained depends upon the volume of the bubble. If a bubble grows while translating at constant velocity, the mass of entrained fluid increases, and a force must be applied to the bubble in order to accelerate the additional mass of entrained fluid. The force on a single bubble growing in unbounded, inviscid fluid has been computed (e.g. Lhuillier, 1982), but no estimates are available when the volume fraction of growing bubbles within a bubbly fluid is large.

Zuber (1964) predicted the added mass of bubbles of fixed radius in a concentrated suspension, using a cell model in which the bubble was at the centre of a fluid-filled sphere. He cites an analysis in Lamb (1932, §93) who in turn refers to earlier work by Stokes. In the frame of zero mixture velocity (i.e. the frame in which the outer spherical boundary is at rest), the

[1] E-mail: sherwood@cambridge.scr.slb.com

PII: S0301-9322(98)00060-3

force F on the gas bubble is

$$F = \frac{\rho_L V_G}{2}\left(\frac{3-2\epsilon_L}{\epsilon_L}\right)\dot{U}_G \tag{1}$$

where ρ_L is the density of the liquid, V_G the volume of the gas bubble, ϵ_L the volume fraction of liquid, U_G the velocity of the bubble and the dot $\dot{}$ represents differentiation with respect to time t, so that $\dot{U}_G$ is the bubble acceleration. Zuber's analysis, in which the outer boundary is at rest, has been extended by Cai and Wallis (1994) in order to examine the effect of the impedance of the external boundary. Although cell models are only approximate, Sangani et al. (1991) and Sangani and Prosperetti (1993) found that Eq. (1) was a good approximation to results obtained by more rigorous analysis.

Here an analysis similar to that of Zuber is performed in order to estimate the forces acting on a bubble which grows: there is no guarantee that the results will be as accurate as they are for a suspension of bubbles of fixed volume.

Lamb's (1932) analysis is based on the total kinetic energy of the liquid between two concentric spheres, the inner of which translates at a fixed velocity $U_G = \dot{\epsilon}$. If the inner sphere is displaced by a distance ϵ, then by symmetry the total kinetic energy varies by at most $O(\epsilon^2)$ rather than $O(\epsilon)$. If the inner sphere accelerates the rate of change of the kinetic energy contains terms $O(\epsilon\dot{\epsilon})$, which are zero when $\epsilon = 0$. An analysis based on spheres which are instantaneously concentric therefore suffices to obtain the added mass. These arguments no longer hold when the inner sphere translates and grows simultaneously. It is therefore safer to use the unsteady form of Bernoulli's equation in order to evaluate explicitly the forces acting on the bubble. We shall require the time derivative of the fluid velocity potential, which must therefore be evaluated to $O(\epsilon)$.

2. Potential flow between two nearly concentric spheres

We assume that the bubble has radius R_1. The outer container, of radius R_2, is filled with incompressible, inviscid fluid of density ρ_L. We assume irrotational flow, and hence seek a velocity potential ϕ such that the fluid velocity

$$u = \nabla\phi \tag{2}$$

where

$$\nabla^2\phi = 0. \tag{3}$$

We use a spherical polar coordinate system (r, θ) based on the centre of the container, and assume that the centre of the bubble is at $\epsilon\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is a unit vector in the direction $\theta = 0$, with $\epsilon \ll R_1$ and $\epsilon \ll R_2 - R_1$. The surface of the bubble is

$$r = R_1 + \epsilon\cos\theta \tag{4}$$

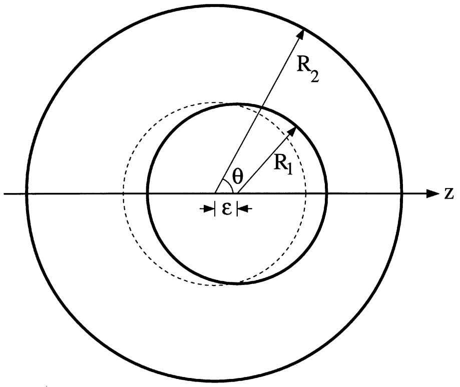

Fig. 1. The sphere of gas with radius R_1 and centre at $z = \epsilon$, surrounded by liquid, with outer spherical boundary of radius R_2 and centre at $z = 0$.

with normal

$$n = \left(1, \frac{\epsilon \sin\theta}{R_1}\right) \tag{5}$$

to leading order in ϵ/R_1. The geometry is shown in Fig. 1. We look for a velocity potential of the form

$$\phi = \frac{A}{r} + \left(Br + \frac{C}{r^2}\right)\cos\theta + \left(Dr^2 + \frac{E}{r^3}\right)(3\cos^2\theta - 1) + \cdots. \tag{6}$$

Incompressibility requires that the radius of the outer container changes when the bubble grows. We assume that both the bubble and outer container remain spherical, with the volume of liquid $\frac{4}{3}\pi(R_2^3 - R_1^3)$ constant: the liquid volume fraction ϵ_L and gas volume fraction ϵ_G are related by

$$\epsilon_L = 1 - (R_1/R_2)^3 = 1 - \epsilon_G. \tag{7}$$

On equating the velocity of the outer boundary to that of the liquid normal to the boundary, we find

$$A = -R_2^2\dot{R}_2 = -R_1^2\dot{R}_1 \tag{8}$$

$$B = 2C/R_2^3 \tag{9}$$

$$2DR_2 = 3E/R_2^4. \tag{10}$$

The bubble translates with velocity $U_G = \dot{\epsilon}$ along the direction $\theta = 0$ so that the total velocity normal to the bubble surface is

$$U_G^n = \dot{R}_1 + \dot{\epsilon}\left(\cos\theta - \frac{\epsilon\sin^2\theta}{R_1}\right) + O(\epsilon^2) \tag{11}$$

whereas the fluid velocity normal to the surface of the bubble given by Eq. (6) is

$$u^n = \left[\frac{\partial\phi}{\partial r} + \left(\frac{\epsilon\sin\theta}{R_1}\right)\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right]_{r=R_1+\epsilon\cos\theta} + O(\epsilon^2). \tag{12}$$

Comparing Eqs. (11) and (12), we obtain, after some algebra

$$B = -\frac{R_1^3[\dot\epsilon + 2\epsilon(\dot R_1/R_1)]}{R_2^3 - R_1^3} \tag{13}$$

$$C = -\frac{R_1^3 R_2^3[\dot\epsilon + 2\epsilon(\dot R_1/R_1)]}{2(R_2^3 - R_1^3)} \tag{14}$$

$$D = \frac{3\epsilon\dot\epsilon R_2^3 R_1^3}{4(R_1^5 - R_2^5)(R_2^3 - R_1^3)} \tag{15}$$

with E given by Eq. (10). Note that D and E are $O(\epsilon)$, and vanish when $\epsilon = 0$.

We shall determine the forces acting on the outer spherical boundary, and on a bubble concentric with the outer boundary, i.e. when $\epsilon = 0$. This will require the unsteady form of Bernoulli's equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho_L} = C_b \tag{16}$$

where p is the fluid pressure and, for irrotational flow, C_b is a constant throughout the fluid.

On the outer boundary

$$u_r = -A/R_2^2 \tag{17}$$

$$u_\theta = -\left(B + \frac{C}{R_2^3}\right)\sin\theta - \left(DR_2 + \frac{E}{R_2^3}\right)6\sin\theta\cos\theta + O(\epsilon^2) \tag{18}$$

and

$$\frac{\partial\phi}{\partial t} = \frac{\dot A}{R_2} + \left(\dot B R_2 + \frac{\dot C}{R_2^2}\right)\cos\theta + \left(\dot D R_2^2 + \frac{\dot E}{R_2^3}\right)(3\cos^2\theta - 1) + \cdots. \tag{19}$$

The force F_2 acting on the outer boundary in the direction $\theta = 0$ is

$$F_2 = 2\pi R_2^2 \int_0^\pi p(R_2, \theta)\cos\theta\sin\theta\,\mathrm{d}\theta \tag{20}$$

and hence, when $\epsilon = 0$

$$F_2 = -\frac{4\pi R_2^3 \rho_L}{3}\left(\dot{B} + \frac{\dot{C}}{R_2^3}\right)$$

$$= \frac{4}{3}\pi R_2^3 \rho_L \left[\frac{3\ddot{\epsilon} R_1^3}{2(R_2^3 - R_1^3)} + \frac{\dot{\epsilon}\dot{R}_1 R_1^2 (15R_2^3 + 3R_1^3)}{2R_2^3(R_2^3 - R_1^3)}\right]. \tag{21}$$

The mean pressure gradient is

$$\frac{\partial p}{\partial z} = F_2\left(\frac{4}{3}\pi R_2^3\right)^{-1}$$

$$= \frac{\rho_L}{2(R_2^3 - R_1^3)}\left[3\ddot{\epsilon} R_1^3 + \frac{\dot{\epsilon}\dot{R}_1 R_1^2 (15R_2^3 + 3R_1^3)}{R_2^3}\right] \tag{22}$$

where z is the direction $\theta = 0$. On the surface of the bubble, when $\epsilon = 0$

$$u_r = \dot{R}_1 + \dot{\epsilon}\cos\theta. \tag{23}$$

u_θ and $\partial\phi/\partial t$ are given by expressions analogous to Eqs. (18)–(19). The force F_1 acting on the bubble in the direction $\theta = 0$ is

$$F_1 = -2\pi R_1^2 \int_0^\pi p(R_1, \theta)\cos\theta \sin\theta \, d\theta \tag{24}$$

and hence, when $\epsilon = 0$

$$F_1 = \frac{4\pi R_1^3 \rho_L}{3}\left(\dot{B} + \frac{\dot{C}}{R_1^3} + \frac{\dot{\epsilon}\dot{R}_1}{R_1}\right)$$

$$= -\frac{4}{3}\pi R_1^3 \rho_L \left\{\frac{\ddot{\epsilon}(2R_1^3 + R_2^3)}{2(R_2^3 - R_1^3)} + \frac{\dot{\epsilon}\dot{R}_1(15R_1^3 + 3R_2^3)}{2R_1(R_2^3 - R_1^3)}\right\}. \tag{25}$$

Note that the total external force which must be applied to the system is

$$-(F_1 + F_2) = -\frac{4}{3}\pi R_1^3 \rho_L \left[\ddot{\epsilon} + \frac{6\dot{R}_1\dot{\epsilon}}{R_1}\right]. \tag{26}$$

The pressure p varies over the surface of the bubble, and we have assumed that surface tension is sufficiently strong to keep the bubble close to spherical. The mean pressure p_S over the bubble surface when $\epsilon = 0$ is

$$\frac{p_S}{\rho_L} = C_b - \frac{\dot{A}}{R_1} - \frac{1}{2}\dot{R}_1^2 - \frac{1}{6}\dot{\epsilon}^2 - \frac{1}{3}\left(B + \frac{C}{R_2^3}\right)^2. \tag{27}$$

The mean pressure p_L over the volume of liquid V_L when $\epsilon = 0$ is

$$\frac{4}{3}\pi(R_2^3 - R_1^3)p_L = 2\pi \int_{R_1}^{R_2} r^2 \, dr \int_0^{\pi} p(r, \theta)\sin\theta \, d\theta$$

$$= 2\pi\rho_L \int_{R_1}^{R_2} \left[2C_b - \frac{2\dot{A}}{r} - \frac{A^2}{r^4} - \frac{1}{3}\left(B - \frac{2C}{r^3}\right)^2 - \frac{2}{3}\left(B + \frac{C}{r^3}\right)^2 \right] r^2 \, dr. \tag{28}$$

Hence

$$\frac{p_L - p_S}{\rho_L} = \frac{R_2^3 \dot{\epsilon}^2}{4(R_2^3 - R_1^3)} - \frac{3\dot{R}_1^2 (R_2 - R_1)^2 (R_2^2 - R_1^2)}{2R_2(R_2^3 - R_1^3)} - \frac{R_1 \ddot{R}_1 (R_2 - R_1)^2 (2R_2 + R_1)}{2(R_2^3 - R_1^3)}. \tag{29}$$

The mean pressure gradient $(\nabla p)_L$ within the liquid volume V_L is

$$\frac{4}{3}\pi(R_2^3 - R_1^3)\overline{\nabla(p)_L} = \int_{V_L} (\nabla p) \, dV = \int_{S_L} p \, d\mathbf{n} \tag{30}$$

where $\mathbf{n}$ is the outward facing normal of the liquid surface S_L, and hence when $\epsilon = 0$

$$\left(\frac{\partial p}{\partial z}\right)_L = \frac{3(F_1 + F_2)}{4\pi(R_2^3 - R_1^3)} = \frac{R_1^3 \rho_L}{(R_2^3 - R_1^3)} \left[\ddot{\epsilon} + \frac{6\dot{R}_1 \dot{\epsilon}}{R_1} \right]. \tag{31}$$

If $\dot{R}_1 = 0$, then by Eq. (25) the force on the bubble is

$$F_1 = -\frac{4}{3}\pi R_1^3 \rho_L \dot{U}_G \left(\frac{1 + 2\epsilon_G}{1 - \epsilon_G} \right) \tag{32}$$

in agreement with Zuber (1964). As $R_2 \rightarrow \infty$

$$F_1 \rightarrow -\frac{4}{3}\pi R_1^3 \rho_L \left[\frac{\ddot{\epsilon}}{2} + \frac{3\dot{R}_1 \dot{\epsilon}}{2R_1} \right] \tag{33}$$

and by Eq. (29)

$$\frac{p_S}{\rho_L} \rightarrow \frac{p_L}{\rho_L} - \frac{\dot{\epsilon}^2}{4} + \frac{3\dot{R}_1^2}{2} + R_1 \ddot{R}_1 \tag{34}$$

as found by Lhuillier (1982) in unbounded fluid.

Since the mixture velocity is zero, we may define a mean liquid velocity U_L such that

$$(R_2^3 - R_1^3)U_L + R_1^3 U_G = 0. \tag{35}$$

It is tempting to differentiate Eq. (35) in order to obtain the mass-acceleration of the liquid, and hence that of the entire system (on the assumption that the density of the gas is zero). This leads to

$$\frac{4}{3}\pi\rho_L (R_2^3 - R_1^3)\dot{U}_L = -\frac{4}{3}\pi\rho_L [R_1^3 \dot{U}_G + 3R_1^2 \dot{R}_1 U_G. \tag{36}$$

However, this argument is wrong. We see that although the term in $\ddot{\epsilon} = \dot{U}_G$ in expression (36) for the total mass-acceleration agrees with the corresponding term in the total force, Eq. (26), the terms in $\dot{\epsilon}\dot{R}$ do not agree. We must be more careful. Let us assume that the centre of mass of the liquid is at $z = \delta$ and that of the gas is at $z = \epsilon$. Then

$$(R_2^3 - R_1^3)\delta + R_1^3\epsilon = 0 \tag{37}$$

so that

$$(R_2^3 - R_1^3)\dot{\delta} + R_1^3\dot{\epsilon} + 3R_1^2\dot{R}_1\epsilon = 0 \tag{38}$$

and

$$(R_2^3 - R_1^3)\ddot{\delta} + R_1^3\ddot{\epsilon} + 6R_1^2\dot{R}_1\dot{\epsilon} + (6R_1\dot{R}_1^2 + 3R_1^2\ddot{R}_1)\epsilon = 0. \tag{39}$$

Thus, when $\epsilon = 0$, the mass-acceleration given by Eq. (39) does indeed agree with the total force, Eq. (26). The velocity $\dot{\delta}$ of the centre of mass is equal to U_L when $\dot{\epsilon}=0$, as shown by Eq. (38), but the acceleration $\ddot{\delta}\neq\dot{U}_G$ if $\dot{R}_1\neq 0$.

3. Application to two-fluid models for multiphase flow

We now consider how the results obtained above may be used in two-fluid models (in some approximate way) when the disperse phase volume fraction is not small. We first consider incompressible bubbles. Zhang and Prosperetti (1994a) studied dilute suspensions of spherical particles, and obtained, by rigorous analysis, equations for the motion of the continuous and disperse phases. We write their Eq. (5.10) for the disperse (gas) phase in the form

$$\begin{aligned}\rho_G\epsilon_G\left[\frac{\partial\langle\mathbf{u}_G\rangle}{\partial t} + \langle\mathbf{u}_G\rangle\cdot\nabla\langle\mathbf{u}_G\rangle\right] = &-\epsilon_G\nabla\langle p_L\rangle \\ &+\frac{C_1\rho_L\epsilon_G}{2}\left[\frac{\partial\langle\mathbf{u}_L\rangle}{\partial t} + \langle\mathbf{u}_L\rangle\cdot\nabla\langle\mathbf{u}_L\rangle - \frac{\partial\langle\mathbf{u}_G\rangle}{\partial t} - \langle\mathbf{u}_G\rangle\cdot\nabla\langle\mathbf{u}_G\rangle\right] \\ &+\frac{1}{2}\rho_L(\nabla\wedge\langle\mathbf{u}_L\rangle)\wedge(\langle\mathbf{u}_G\rangle - \langle\mathbf{u}_L\rangle) \\ &+\left(\rho_G + \frac{1}{2}\rho_L\right)\nabla\cdot(\epsilon_G\mathbf{M}_d) + \epsilon_G\mathbf{F}_G.\end{aligned} \tag{40}$$

Here ρ_G is the density of the gas, $\langle\mathbf{u}_G\rangle$ and $\langle\mathbf{u}_L\rangle$ are the ensemble averages of the velocity in the gas and liquid phases, $\langle p_L\rangle$ is the ensemble average of the pressure in the continuous (liquid) phase, the added mass coefficient $C_1 = 1$ in a dilute suspension, $\mathbf{M}_d$ is the fluctuating volume flux tensor and $\mathbf{F}_G$ is the force per unit volume acting on the bubble, which Zhang and Prosperetti (1994a) took to be that due to gravity. We assume that the flow is irrotational so that the third term on the right-hand side of Eq. (40) is zero. The cell model of Section 2 has only one bubble, so fluctuations are zero and we set $\mathbf{M}_d = 0$. We now apply the momentum Eq. (40) to motion in the concentric cell model of Section 2 in the direction $\theta = 0$, and identify the ensemble average velocities of Eq. (40) with the mean velocities of Eq. (35), so that $\langle \mathbf{u}_G \rangle = U_G = \dot{\epsilon}$ and $\langle \mathbf{u}_L \rangle = U_L = -R_1^3\dot{\epsilon}/(R_2^3 - R_1^3)$. The gas density $\rho_G = 0$. The external force acting on the bubble is $F_G = -F_1$, where F_1 is given by Eq. (25). The continuous phase average pressure $\langle p_L \rangle$ is defined by Zhang and Prosperetti in such a way that differentiation does not commute with averaging: we nevertheless assume that $\nabla \langle p_L \rangle$ is given by the mean pressure gradient $(\nabla p)_L$, Eq. (31). The acceleration of the (incompressible) bubble is $\ddot{\epsilon}$, and the acceleration of the liquid is $\ddot{\delta} = -R_1^3\ddot{\epsilon}/(R_2^3 - R_1^3)$ by (39). Inserting all these results into Eq. (40) we obtain

$$0 = -\frac{\rho_L R_1^3 \ddot{\epsilon}}{R_2^3 - R_1^3} - \frac{C_1 \rho_L R_2^3 \ddot{\epsilon}}{2(R_2^3 - R_1^3)} + \frac{\rho_L \ddot{\epsilon}(2R_1^3 + R_2^3)}{2(R_2^3 - R_1^3)} \tag{41}$$

and hence

$$C_1 = 1. \tag{42}$$

Zhang and Prosperetti (1994a, Sections 6 and 7) discuss the extension of their results to linear problems at higher volume fractions, and C_1 in Eqs. (40)–(41) is equivalent to their C. They conclude that previously published results suggest that C_1 is close to 1, with Zuber's added mass (1) corresponding to $C_1 = 1$. Zuber's added mass contains a contribution to the force on the bubble due to the pressure gradient: when the pressure gradient is explicitly included in governing equations of the form (40) it is important that this contribution should not be included twice due to incorrect choice of C_1.

In a second paper Zhang and Prosperetti (1994b) consider spherical, compressible bubbles. If all the bubbles have the same size, in the absence of particle inertia, fluctuations and vorticity Zhang and Prosperetti's disperse phase momentum Eq. (99) (equivalent to Eq. (40) above) becomes

$$\epsilon_G \nabla \langle p_L \rangle = \frac{C_1 \rho_L \epsilon_G}{2}\left[\frac{\partial \langle \mathbf{u}_L \rangle}{\partial t} + \langle \mathbf{u}_L \rangle \cdot \nabla \langle \mathbf{u}_L \rangle - \frac{\partial \langle \mathbf{u}_G \rangle}{\partial t} - \langle \mathbf{u}_G \rangle \cdot \nabla \langle \mathbf{u}_G \rangle\right]$$

$$+ \frac{1}{2} C_2 \rho_L n \dot{V}_G(\langle \mathbf{u}_L \rangle - \langle \mathbf{u}_G \rangle) + \epsilon_G \mathbf{F}_G \tag{43}$$

where $C_1 = C_2 = 1$ in a dilute suspension, and n is the number density of bubbles so that nV_G is (to a first approximation) the gas volume fraction ϵ_G. The compressible momentum Eq. (43) differs from the incompressible Eq. (40) by the term in $\dot{V}_G$, and nothing is known about how the coefficient C_2 of this term varies with volume fraction.

If we follow the same substitutions for the forces and velocities as above, but keep terms in $\dot{\epsilon}\dot{R}_1$, we find that Eq. (43) is identically satisfied if

$$C_1 = 1, \quad C_2 = 1 - \epsilon_G. \tag{44}$$

This suggests how equations of the form (43) should be modified when the dispersed volume fraction is no longer small, though there are as yet no computations (equivalent to those of Sangani et al. (1991) for incompressible bubbles) to give rigorous justification for such an extension to large volume fractions.

References

Cai, X., Wallis, G.B., 1994. A more general cell model for added mass in two-phase flow. Chem. Engng Sci. 49, 1631–1638.

Lamb, H., 1932. Hydrodynamics. 6th ed. Cambridge University Press, Cambridge.

Lhuillier, D., 1982. Forces d'inertie sur une bulle en expansion se déplaçant dans un fluide. Comptes Rendus Acad. Sci. Paris Sér. II 295, 95–98.

Sangani, A.S., Prosperetti, A., 1993. Numerical simulation of the motion of particles at large Reynolds numbers. In: M.C. Roco (Ed.). Particulate Two-phase Flow. Butterworth-Heinemann, Boston, MA, pp. 971–998.

Sangani, A.S., Zhang, D.Z., Prosperetti, A., 1991. The added mass, Basset, and viscous drag coefficients in non-dilute bubbly liquids undergoing small-amplitude oscillatory motion. Phys. Fluids A 3, 2955–2970.

Zhang, D.Z., Prosperetti, A., 1994a. Averaged equations for inviscid disperse two-phase flow. J. Fluid Mech. 267, 185–219.

Zhang, D.Z., Prosperetti, A., 1994b. Ensemble phase-averaged equations for bubbly flows. Phys. Fluids 6, 2956–2970.

Zuber, N., 1964. On the dispersed two-phase flow in the laminar-flow regime. Chem. Engng Sci. 19, 897–917.